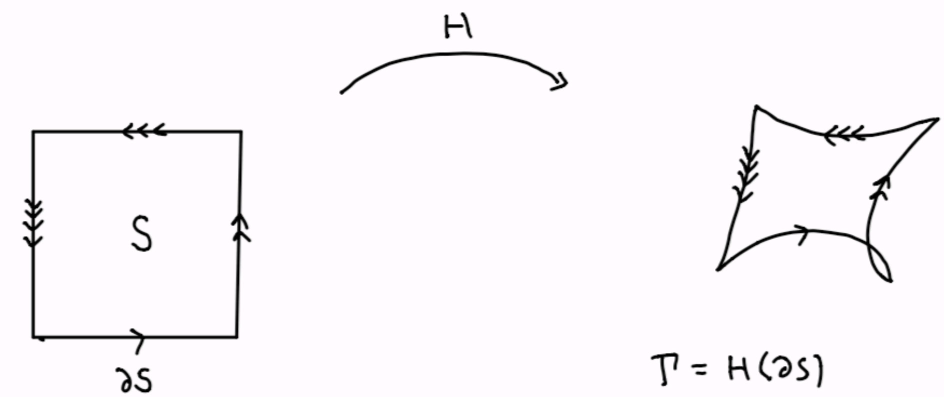
From Friday:

<u>Homotopy Lemma</u> Let $A \subseteq \mathbb{C}$ be open and connected. Let $f: A \to \mathbb{C}$ be analytic. Let

$$S = \{ (s, t) \mid 0 \le s \le 1,\ 0 \le t \le 1 \} \text{ and}$$
$$\delta S$$

denote the unit square and its boundary, oriented counterclockwise. Let $H: S \to A$ be continuous, with $\Gamma := H(\delta S)$ a piecewise C^1 contour. Then

$$\int_{\Gamma} f(z)\, dz = 0.$$

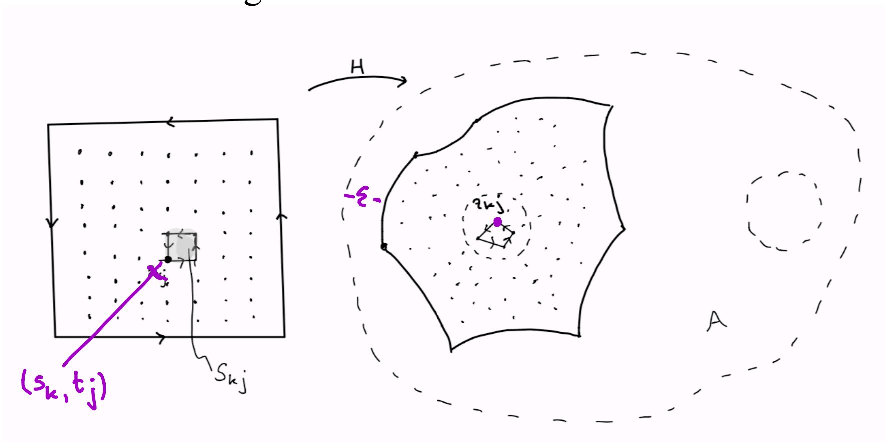


only depends on local antiderivs.

proof of the homotopy lemma: Subdivide S into n^2 subsquares of side lengths n^{-1}. The dots in the diagram on the left indicate their vertices. number the squares as you would a matrix, and let S_{kj} be a typical subsquare, with $\boldsymbol{z}_{kj}$ be the image under the homotopy of its lower left corner. Since H is continuous and S is compact, the image $H(S) \subseteq A$ is compact. Write

$$H(\delta S) = \Gamma$$
$$H(\delta S_{kj}) = \Gamma_{kj}.$$

Replace any of the four subarcs of each Γ_{kj} which are not C^1 with constant speed line segment paths between the image vertices.



By interior cancellation,

$$\int_{\Gamma} f(\boldsymbol{z})\, d\boldsymbol{z} = \sum_{k,j} \int_{\Gamma_{kj}} f(\boldsymbol{z})\, d\boldsymbol{z}.$$

Note:

1) $H(S)$ is compact, $H(S) \subseteq A$ open, so by the Positive Distance Lemma you're proving in this week's homework

$$\exists\, \varepsilon > 0 \text{ such that } \forall\, \boldsymbol{z} \in H(S),\ D(\boldsymbol{z}; \varepsilon) \subseteq A.$$

2) H is continuous on S so H is uniformly continuous. Thus for ε as in (1),

$$\exists\, \delta > 0 \text{ such that } \left\| (s, t) - \left(\tilde{s}, \tilde{t}\right) \right\| < \delta \Rightarrow \left| H(s, t) - H\left(\tilde{s}, \tilde{t}\right) \right| < \varepsilon.$$

3) If n is large enough so that the diagonal length of the subsquares is less than δ, then each

$$H(S_{kj}) \subseteq D(\boldsymbol{z}_{kj}; \varepsilon) \subseteq A,\ \boldsymbol{z}_{kj} = H(s_k, t_j).$$

4) By the local antidifferentiation theorem in $D(\boldsymbol{z}_{kj}; \varepsilon)$, each

$$\int_{\Gamma_{kj}} f(\boldsymbol{z})\, d\boldsymbol{z} = 0 \Rightarrow \int_{\Gamma} f(\boldsymbol{z})\, d\boldsymbol{z} = 0. \qquad \text{Q.E.D.!!!}$$

Math 4200
Wednesday October 7.
2.4: We have a bit more to discuss in Monday's notes, and a fun quick theorem in today's. After that we'll discuss our first midterm on Friday.

Announcements:

- Midterm Fri
- All HW sols are (or will) be posted
  all except 1 & 2 will be graded before exam.
- @ 2 p.m Thurs I'll go over an old exam (video)
  & I'll post sol'ns /comments.

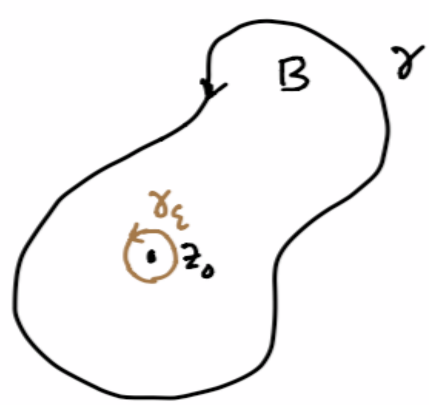
Math 4200-001
Week 7-8 concepts and homework
2.4
Due Friday October 16 at 11:59 p.m.

2.4 2, 3, 5, 7, 8, 12, 16, 17, 18. Hint: In problems 2, 5, 18 identify the contour integrals as expressing a certain function or one of its derivatives, at a point inside γ, via the Cauchy integral formulas for analytic functions and their derivatives.

w7.1 Prove the special case of the Cauchy integral formula that we discuss on Monday/Wednesday.
we discussed.

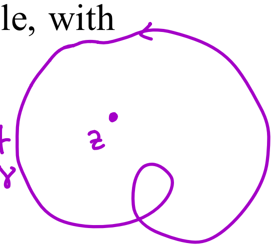
If γ is a counter-clockwise simple closed curve bounding a subdomain B in A, with z_0 inside γ, then the important special case of the Cauchy integral formula can be proven with contour replacement and a limiting argument, assuming f is C^1 in addition to being analytic:

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz.$$



First application of C.I.F. : f analytic implies f is infinitely differentiable, with estimates for the moduli of the derivatives. Rewrite the CIF as

$I(\gamma;z)$ is locally constant inside γ



$\frac{d}{dz}$: $\quad f'(z)\,I(\gamma;z) = \frac{1}{2\pi i}\int_\gamma \frac{d}{dz}\frac{f(\zeta)}{\zeta - z}\,d\zeta$

$$f(z)I(\gamma\,;z) = \frac{1}{2\,\pi\,i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

(we replaced z_0 by z, and the contour integral variable z by ζ.)

Theorem 1 Let f be analytic in the open set $A \subseteq \mathbb{C}$, γ a p.w. C^1 contour homotopic to a point in A. Then for z inside γ, every derivative of f exists and may be computed by the contour integral formulas

- $$f'(z)I(\gamma\,;z) = \frac{1}{2\,\pi\,i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta$$
- $$f^{(n)}(z)I(\gamma\,;z) = \frac{n!}{2\,\pi\,i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$

notice, these are the formulas we get by induction and "differentiating thru the integral sign" :

$$\left[\ \frac{d}{dz}\frac{f(\zeta)}{\zeta - z} = f(\zeta)(-1)(\zeta - z)^{-2}(-1) = \frac{f(\zeta)}{(\zeta - z)^2}\ \right]$$

$$\frac{d}{dz}\frac{f(\zeta)}{(\zeta - z)^n} = f(\zeta)(-n)(\zeta - z)^{-n-1}(-1) = n\frac{f(\zeta)}{(\zeta - z)^{n+1}}.$$

So, when can you justify this operation of differentiating thru the integral sign? That's an analysis question for after the midterm!

<u>Corollary</u> *(Liouville's Theorem)* Let $f:\mathbb{C}\to\mathbb{C}$ be entire. Suppose f is also *bounded*, i. e. $\exists\ M\in\mathbb{R}$ such that $|f(z)|\le M\ \ \forall\ z\in\mathbb{C}$. Then f is constant.

proof: (It's very very short.)

Let $z \in \mathbb{C}$.

Let γ : circle of rad R ctrd @ z

$$f'(z) = \frac{1}{2\pi i}\oint_{|\zeta-z|=R} \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta .$$

$$|f'(z)| \le \frac{1}{2\pi}\oint_{|\zeta-z|=R} \frac{M}{R^2}\,|d\zeta| = \frac{1}{2\pi}\frac{M}{R^2}\cdot 2\pi R = \frac{M}{R}.$$

f entire, so R is arb. large.

$\lim_{R\to\infty}$: $\quad |f'(z)| \le 0 \quad \Rightarrow f' \equiv 0 \Rightarrow f$ is const!

Exam Friday October 9

email me (korevaar "at" math.utah.edu) your preferred two hour time slot on Friday, starting on the hour between 10:00 a.m. and 4:00 p.m.

I'll give you my cell # on CANVAS, text me if issues.

I will email you a .pdf of the exam at the start time or a few minutes early. Unless you tell me otherwise I'll email it to the return address of the email you send me.

Complete the exam and upload a .pdf of your solutions to Gradescope by two hours after your start time. For insurance or if you have trouble uploading, email me a .pdf within your time limit as well.

The exam is closed book, closed notes, closed internet etc. Your only resource is yourself. I'll ask you to sign an honor-code like statement on the front of your exam which will be part of what you upload to Gradescope.

Potential Topics (we'll discuss): I've uploaded practice test — CANVAS.

Complex differentiability (def at a point, equivalent approximation formula, and "analytic" on a domain).

Cauchy Riemann equations $u_x = v_y$, $u_y = -v_x$

relationship to real differentiability, i.e. $f: \mathbb{C} \to \mathbb{C}$ analytic at z_0 is equivalent to what for $F: \mathbb{R}^2 \to \mathbb{R}^2$ at (x_0, y_0)?

$f(x+iy) = u(x,y) + i\,v(x,y)$

"(u,v)" iff F is real diff'ble & CR hold.

consequences of definition of derivative and equivalent approximation formula

- sum, product, quotient rules
- chain rule
- chain rule for curves
- differential map df_{z_0}

- using chain rule for curves to write CR in different coordinate systems.

- inverse function theorem

harmonic functions and harmonic conjugates in simply-connected domains

brackets = more likely topics

Complex transformations

polar form for complex multiplication, powers, exponentials, logarithms.

$f(z) = az + b$, z^n, e^z, $\log z$, z^a, $\cos z$, $\sin z$, compositions

branch points, branch cuts, branch domains for root functions, logarithms, and compositions most probably!

Contour integration

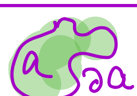
definition and computation

relation to real-variables line integrals

$$\int_\gamma (u(x,y) + i v(x,y))(dx + i\,dy)$$

$$= \int_\gamma u\,dx - v\,dy + i \int_\gamma v\,dx + u\,dy$$

Green's Theorem for contour integrals around domains (including domains with holes).

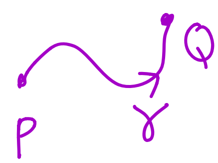


$$\int_{\partial a} P\,dx + Q\,dy = \iint_a \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

contour replacement for C^1 analytic integrands $f(z)$, via Green's Theorem and CR equations (Section 2.2 Cauchy's Theorem and Deformation/Replacement Theorem.)

estimates for modulus of contour integrals.

FTC $\int_\gamma f(z)\,dz = F(Q) - F(P)$



evaluation of contour integrals when the integrand is analytic, using FTC and/or contour replacement.

§ 2.2.

<u>Homotopy-related ideas</u>

homotopies

fixed endpoint

of closed paths

simply-connected domains

<u>Antiderivatives of analytic functions</u>

equivalence to path independence

§2.3 local anti-derivative theorem, using rectangle lemma

§2.3 global antiderivatives in (open) simply-connected domains, using homotopy lemma to prove path independence

<u>Deformation Theorems</u> via the homotopy lemma

for contours with fixed endpoints

for closed curves (section 2.3 Cauchy's Theorem)